On Lifting-Based Fixed-Point Complex Multiplications and Rotations

Oscar Gustafsson

LINKÖPING UNIVERSITY

Complex Multiplications and Rotations

- Consider a complex multiplication with $e^{j\theta}$ or a plane rotation with an angle θ

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{1}$$

where $c = \cos(\theta)$ and $d = \sin(\theta)$

Complex Multiplications and Rotations

- Consider a complex multiplication with $e^{j\theta}$ or a plane rotation with an angle θ

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{1}$$

  where $c = \cos(\theta)$ and $d = \sin(\theta)$
- Denote the fixed-point approximation of a continuous variable x as $\hat{x}$

Complex Multiplications and Rotations

- Consider a complex multiplication with $e^{j\theta}$ or a plane rotation with an angle θ

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{1}$$

  where $c = \cos(\theta)$ and $d = \sin(\theta)$

- Denote the fixed-point approximation of a continuous variable x as $\hat{x}$
- For (1) we expect $\hat{c}^2 + \hat{d}^2 = 1$

Complex Multiplications and Rotations

- If $\hat{c}$ and $\hat{d}$ are represented in a binary (base-2) fixed-point representation this will never happen (unless $\hat{c}, \hat{s} \in \{0, 1\}$)

Complex Multiplications and Rotations

- If $\hat{c}$ and $\hat{d}$ are represented in a binary (base-2) fixed-point representation this will never happen (unless $\hat{c}, \hat{s} \in \{0, 1\}$)
- Assuming a B-bit representation, we define $C = \hat{c}2^B$, i.e., C is a integer
- Then, $C^2 + D^2 = 2^{2B} = Z^2$

Complex Multiplications and Rotations

- If $\hat{c}$ and $\hat{d}$ are represented in a binary (base-2) fixed-point representation this will never happen (unless $\hat{c}, \hat{s} \in \{0, 1\}$)
- Assuming a B-bit representation, we define $C = \hat{c}2^B$, i.e., C is a integer
- Then, $C^2 + D^2 = 2^{2B} = Z^2$
- It can be shown that Z must be on the form $4N + 1$ for the equality to have integer solutions, and, hence, $\hat{c}^2 + \hat{d}^2 \neq 1$

Complex Multiplications and Rotations

- This leads to that it is not possible to rotate a value exactly back to the original
- Consider the rotation matrix $\mathbf{R}$

$$\mathbf{R} = \begin{bmatrix} \hat{c} & -\hat{d} \\ \hat{d} & \hat{c} \end{bmatrix}. \quad (2)$$

- The inverse is then equal to

$$\mathbf{R}^{-1} = \frac{1}{\det R} \begin{bmatrix} \hat{c} & \hat{d} \\ -\hat{d} & \hat{c} \end{bmatrix} = \frac{1}{\hat{c}^2 + \hat{d}^2} \begin{bmatrix} \hat{c} & \hat{d} \\ -\hat{d} & \hat{c} \end{bmatrix}$$

Complex Multiplications and Rotations

- Ignoring that $\hat{c}^2 + \hat{d}^2 \neq 1$ and assigning

$$\mathbf{R}^{-1} = \begin{bmatrix} \hat{c} & \hat{d} \\ -\hat{d} & \hat{c} \end{bmatrix} \tag{3}$$

results in a forward-backward rotation as

$$\mathbf{R}\mathbf{R}^{-1} = \begin{bmatrix} \hat{c} & -\hat{d} \\ \hat{d} & \hat{c} \end{bmatrix} \begin{bmatrix} \hat{c} & \hat{d} \\ -\hat{d} & \hat{c} \end{bmatrix} = \begin{bmatrix} \hat{c}^2 + \hat{d}^2 & 0 \\ 0 & \hat{c}^2 + \hat{d}^2 \end{bmatrix} \tag{4}$$

- Hence, there is a magnitude error in a forward/backward rotation

Integer/Lossless Transforms

- This is an issue in e.g. transform based source coding where losslessness is expected
- Transform based coding are based on e.g. discrete cosine transforms, discrete Fourier transforms, or Wavelet transforms

Lifting-Based Complex Multiplier/Rotator

- One solution is to split $\mathbf{R}$ into three subsequent matrix multiplications
- For example, as

$$\mathbf{R} = \begin{bmatrix} 1 & \frac{c-1}{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{c-1}{d} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix}, \tag{5}$$

  where $g = \frac{c-1}{d}$
- Now,

$$\mathbf{R}^{-1} = \begin{bmatrix} 1 & -g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -d & 1 \end{bmatrix} \begin{bmatrix} 1 & -g \\ 0 & 1 \end{bmatrix} \tag{6}$$

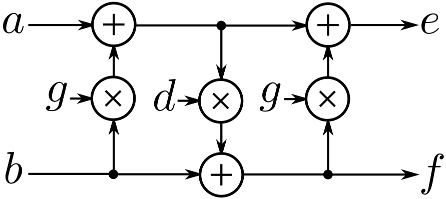
Lifting-Based Complex Multiplier/Rotator

- Applying the forward and backward rotation will always result in the identity matrix as (note that nothing is assumed about d and g here)

$$\begin{aligned}
\mathbf{R}\mathbf{R}^{-1} &= \begin{bmatrix}1 & g\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\d & 1\end{bmatrix}\begin{bmatrix}1 & g\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -g\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\-d & 1\end{bmatrix}\begin{bmatrix}1 & -g\\0 & 1\end{bmatrix}\\
&= \begin{bmatrix}1 & g\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\d & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\-d & 1\end{bmatrix}\begin{bmatrix}1 & -g\\0 & 1\end{bmatrix}\\
&= \dots\\
&= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \qquad (7)
\end{aligned}$$

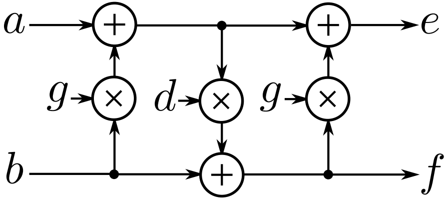
Lifting-Based Complex Multiplier/Rotator

- The lifting-based multiplier can be realized using three multiplications and three additions as



Lifting-Based Complex Multiplier/Rotator

- The lifting-based multiplier can be realized using three multiplications and three additions as



- This structure may also be attractive for low-complexity implementation and when using MAC-operations

Lifting-Based Complex Multiplier/Rotator

- Consider realizing $\mathbf{R}$ with quantized coefficients

$$\mathbf{R} = \begin{bmatrix} 1 & \hat{g} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \hat{d} & 1 \end{bmatrix} \begin{bmatrix} 1 & \hat{g} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + \hat{d}\hat{g} & \hat{g}(2 + \hat{d}\hat{g}) \\ \hat{d} & 1 + \hat{d}\hat{g} \end{bmatrix}. \quad (8)$$

- Clearly, inserting $\hat{g} = \frac{c-1}{\hat{d}}$ we obtain

$$1 + \hat{d}\hat{g} = 1 + \hat{d}\frac{c-1}{\hat{d}} = c \quad (9)$$

and

$$\hat{g}\left(2 + \hat{d}\hat{g}\right) = \frac{c-1}{\hat{d}}\left(2 + \hat{d}\frac{c-1}{\hat{d}}\right) = -\hat{d}. \quad (10)$$

Lifting-Based Complex Multiplier/Rotator

- However, $\hat{g}$ is a quantized value, and, hence, only approximating $\frac{c-1}{\hat{d}}$
- Therefore, we can conclude that the resulting rotation can be written as

$$\mathbf{R} = \begin{bmatrix} \hat{h} & -\hat{k} \\ \hat{d} & \hat{h} \end{bmatrix}, \tag{11}$$

  where $\hat{k} = -\hat{g}(2 + \hat{d}\hat{g})$, $\hat{h} = 1 + \hat{d}\hat{g}$, and $\hat{k} \neq \hat{d}$
- Note that if $\hat{d}$ and $\hat{g}$ have B fractional bits, $\hat{k}$ and $\hat{h}$ have $3B$ and $2B$ fractional bits, respectively

Contributions of the Paper

- We show that the resulting magnitude and angle of the lifting-based rotation is input data dependent and derive expressions for this
- We propose a method to design integer invertible rotations directly, i.e., determining $\hat{d}$, $\hat{h}$, and $\hat{k}$ in

$$\mathbf{R} = \begin{bmatrix} \hat{h} & -\hat{k} \\ \hat{d} & \hat{h} \end{bmatrix} \tag{12}$$

Error Analysis

- First, note that

$$\mathbf{R} = \begin{bmatrix} \hat{h} & -\hat{k} \\ \hat{d} & \hat{h} \end{bmatrix} \tag{13}$$

with $\hat{d} \neq \hat{k}$ is not a proper rotation

Error Analysis

- First, note that
$$\mathbf{R} = \begin{bmatrix} \hat{h} & -\hat{k} \\ \hat{d} & \hat{h} \end{bmatrix} \tag{13}$$
with $\hat{d} \neq \hat{k}$ is not a proper rotation
- Hence, for reference, we select a "correct" rotation
$$\mathbf{R}' = \begin{bmatrix} \hat{h} & -\gamma \\ \gamma & \hat{h} \end{bmatrix}, \tag{14}$$
where $\gamma = \sqrt{\hat{d}\hat{k}}$
- This gives magnitude 1 and angle $\arctan\left(\frac{\gamma}{\hat{h}}\right)$

Error Analysis

- Consider rotating the value $n + jo = r\mathrm{e}^{j\alpha}$

$$\begin{bmatrix} l \\ m \end{bmatrix} = \begin{bmatrix} \hat{h} & -\hat{k} \\ \hat{d} & \hat{h} \end{bmatrix} \begin{bmatrix} n \\ o \end{bmatrix} = \begin{bmatrix} \hat{h}r\cos(\alpha) - \hat{k}r\sin(\alpha) \\ \hat{d}r\cos(\alpha) + \hat{h}r\sin(\alpha) \end{bmatrix} \tag{15}$$

- Introduce the difference between $\hat{d}$ and $\hat{k}$ as $\delta = \hat{d} - \hat{k}$

Error Analysis – Magnitude

$$|l + jm|^2 = r^2 \left(1 + \frac{\delta^2}{2} + \right.$$
$$\left. -\delta\sqrt{\frac{\delta^2}{4} + 1}\sin\left(2\alpha + \arctan\left(\frac{\sqrt{\frac{\delta^2}{4} + \gamma^2}}{\hat{h}}\right)\right)\right) \quad (16)$$
$$\approx r^2 \left(1 + \frac{\delta^2}{2} - \delta\sin\left(2\alpha + \arctan\left(\frac{\gamma}{\hat{h}}\right)\right)\right). \quad (17)$$

Sinusoidal error depending on input angle

Error Analysis – Angle

$$
\begin{aligned}
&\arctan\left(\frac{\hat{h}n-\hat{k}o}{\hat{d}n+\hat{h}o}\right)-\arctan\left(\frac{\hat{h}n-\gamma o}{\gamma n+\hat{h}o}\right)\\
&=\arctan\left(\frac{\delta\sin\left(2\alpha-\arctan\left(\frac{\hat{h}}{\gamma}\right)\right)+\left(\sqrt{\delta^2+4\,\gamma^2}-2\gamma\right)\hat{h}}{\delta\sin\left(2\alpha+\arctan\left(\frac{\gamma}{\hat{h}}\right)\right)-\gamma\sqrt{\delta^2+4\gamma^2}-2\hat{h}^2}\right)\\
&\approx\arctan\left(\frac{\delta\sin\left(2\alpha-\arctan\left(\frac{\hat{h}}{\gamma}\right)\right)}{\delta\sin\left(2\alpha+\arctan\left(\frac{\gamma}{\hat{h}}\right)\right)-2}\right)
\end{aligned}
\tag{18}
$$

Improved Design

- Use the integer version of the values so

$$\hat{h}^2 + \hat{d}\hat{k} = 1 \Rightarrow H^2 + DK = 2^{2B}$$

- Set $K = D + \Delta$
- This gives

$$2^{2B} = H^2 + D^2 + D\Delta$$

- Problem is to find integer solutions with small Δ

Improved Design

- Can be rewritten as

$$P^2 + Q^2 = \Delta^2 + S^2 = T \qquad (19)$$

  where $S = 2^{(B+1)}$ and $Q = 2H$ (both even leading to that Q and Δ are odd)
- Determine a value T that can be written as two different sum of two squares
  - Where one of the sum of squares is formed as a power of two and one is a small odd number

Improved Design

- If T is a composite number only containing prime factors of the form $4N+1$ or 2 and at least two factors of the form $4N+1$, there will be at least two different ways to express T as sums of squares
- However, $T \mod 4 \equiv 1$ and all prime factors are of the $4N+1$, since $T = \Delta^2 + S^2$ and $\gcd(\Delta, S) = 1$

Every candidate number $T = 4^{(B+1)} + \Delta^2$ will either be a prime or result in, at least one, integer invertible rotation

Design Example

- Consider a rotation by $45^\circ = \frac{\pi}{4}$ rad
- With $B = 6$, a integer invertible rotation with $\delta = 1$ is

$$\mathbf{R} = \frac{1}{2^6}\begin{bmatrix} 46 & -45 \\ 44 & 46 \end{bmatrix}$$

- Can not be factorized

Design Example

- Consider a rotation by $45^\circ = \frac{\pi}{4}$ rad
- With $B = 6$, a integer invertible rotation with $\delta = 1$ is

$$\mathbf{R} = \frac{1}{2^6}\begin{bmatrix} 46 & -45 \\ 44 & 46 \end{bmatrix}$$

- Can not be factorized
- Factorized approach

$$\begin{bmatrix} 1 & -\frac{27}{64} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{45}{64} & 1 \end{bmatrix}\begin{bmatrix} 1 & -\frac{27}{64} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2881}{4096} & -\frac{188379}{262144} \\ \frac{45}{64} & \frac{2881}{4096} \end{bmatrix}$$

Design Example

- Consider a rotation by $45^\circ = \frac{\pi}{4}$ rad
- With $B = 6$, a integer invertible rotation with $\delta = 1$ is

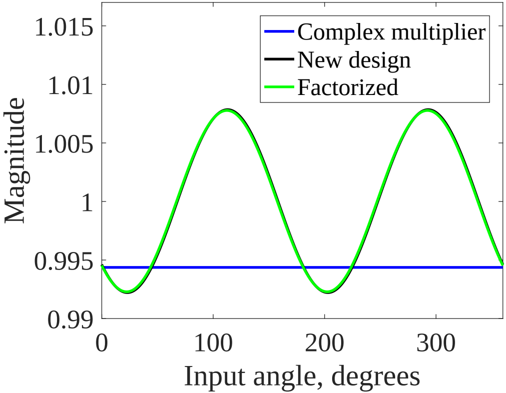
$$\mathbf{R} = \frac{1}{2^6}\begin{bmatrix} 46 & -45 \\ 44 & 46 \end{bmatrix}$$

- Can not be factorized
- Factorized approach

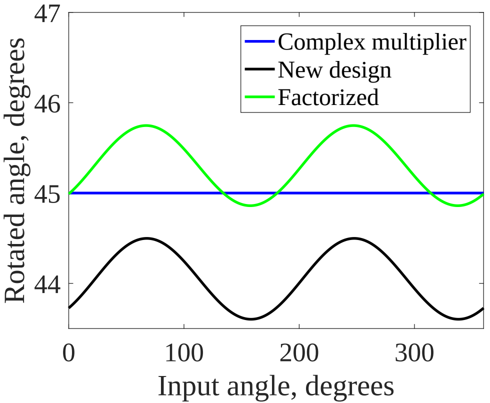
$$\begin{bmatrix} 1 & -\frac{27}{64} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{45}{64} & 1 \end{bmatrix}\begin{bmatrix} 1 & -\frac{27}{64} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2881}{4096} & -\frac{188379}{262144} \\ \frac{45}{64} & \frac{2881}{4096} \end{bmatrix}$$

- Similar δ: $\delta = \frac{1}{64} \approx 0.0156$ and $\delta = \frac{3959}{262144} \approx 0.0151$

Design Example – Amplitude



Design Example – Angle



Possible Rotators

B	Δ	P	Q	H	D/K	$-G$
6	1	127	16	8	63/64	56
		103	76	38	51/52	
		89	92	46	44/45	
	2	112	62	31	55/57	
7	2	32	254	127	15/17	
		152	206	103	75/77	
		184	178	89	91/93	
	3	203	156	78	100/103	

Possible Rotators

- In the paper 59 rotations with up to $B = 3$ and small Δ is listed
- Only 10 of those can be separated, i.e., designed using the old approach

Conclusions

- Lifting-based rotators allow integer invertible/error-free forward-backward rotations
- Potentially with a low-complexity realization

Conclusions

- Lifting-based rotators allow integer invertible/error-free forward-backward rotations
- Potentially with a low-complexity realization
- However, there is an input data dependent error in both angle and magnitude

Conclusions

- Lifting-based rotators allow integer invertible/error-free forward-backward rotations
- Potentially with a low-complexity realization
- However, there is an input data dependent error in both angle and magnitude
- The impact of the error is application dependent

Conclusions

- Lifting-based rotators allow integer invertible/error-free forward-backward rotations
- Potentially with a low-complexity realization
- However, there is an input data dependent error in both angle and magnitude
- The impact of the error is application dependent
- If this error is not a problem, a new design method is proposed

Conclusions

- Lifting-based rotators allow integer invertible/error-free forward-backward rotations
- Potentially with a low-complexity realization
- However, there is an input data dependent error in both angle and magnitude
- The impact of the error is application dependent
- If this error is not a problem, a new design method is proposed
- The design space is significantly expanded

Conclusions

- Lifting-based rotators allow integer invertible/error-free forward-backward rotations
- Potentially with a low-complexity realization
- However, there is an input data dependent error in both angle and magnitude
- The impact of the error is application dependent
- If this error is not a problem, a new design method is proposed
- The design space is significantly expanded
- So far, one must look up suitable rotations from a candidate table...

Thank you!
Questions?

www.liu.se

LINKÖPING UNIVERSITY